

# A Weak Gravity Theorem

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**Abstract:** In a gravitational theory with a massless photon the maximum charge-to-mass ratio of black holes approaches the prediction of the Einstein-Maxwell theory as black hole mass increases:  $Q_{\text{ext}}/M = 1 + \alpha/M^2$  for some constant  $\alpha$ . We will show that  $\alpha > 0$  if below the quantum gravity scale  $\Lambda$  there are many degrees of freedom with a hierarchically small mass gap  $\log(\Lambda/m_{\text{gap}}) \gg 1$ . In this regime one can treat gravity as a non-dynamical background field and derive field-theoretic sum-rules for the coefficients of the leading corrections to the Einstein-Maxwell theory. The positivity of  $\alpha$  follows from the sum-rules. As a consequence, gravitational attraction gets weaker than the electric force among maximally charged black holes as they become lighter, and large extremal black holes can decay into smaller black holes.

## 1 Introduction

It has been conjectured that in any consistent theory of quantum gravity with a massless photon there has to be super-extremal states; states whose charge in Planck units is larger than their mass [1]. This “Weak Gravity Conjecture” (WGC) would imply large extremal black holes decay by emitting super-extremal states.

No contradiction is known to arise if extremal black holes were stable. The evidence for the conjecture comes from explicit string theory compactifications which seem to satisfy the WGC in one way or the other. Moreover WGC fits well within a web of ideas about the unique features of gravitational theories such as those proposed in [2, 3, 4, 5]. Stronger versions of the conjecture, which are more specific about the properties of the proposed super-extremal states, are phenomenologically powerful as they imply nontrivial restrictions on effective field theories. See e.g. [1, 6, 7, 8, 9, 10, 11, 12, 13] for application to models of large-field inflation, and [14, 15, 16] for proposed solutions to the electroweak hierarchy problem. As desirable as it is to have this extra predictivity, in most cases there has been either explicit counter-examples to the strong versions of the conjecture, or ways to model-build around them to avoid a sharp constraint.

Nevertheless the investigation of the WGC has been a fruitful program. For instance, it has motivated a “top-down search” for patterns or counter-examples in string compactifica-

tions. It has become clear that the conjecture has to be generalized to multiple symmetry groups [14, 17], to higher-form symmetries and to dilaton-gravity models [18, 19]. Moreover several intriguing links have been found, for instance, with the cosmic censor conjecture, the super-Planckian displacements in the field space (or absence thereof), and possible emergent nature of gauge symmetries. See e.g. [20, 21, 22, 23, 24, 25]. Via holography the WGC has turned into a concrete statement about the spectrum of conformal field theories [26, 24, 27, 28]. Finally, the question has also been formulated as one about the consistency of the low-energy effective theory of photons and gravitons [29, 30, 31, 32]. A recent review and a more complete list of references can be found in [33].

The idea that small charged black holes might themselves qualify as the conjectured super-extremal states is as old as the conjecture itself. Large extremal black holes have  $Q_{\text{ext}}/M = 1$  in the Einstein-Maxwell theory. However this is an effective theory and as the black hole size shrinks the higher derivative corrections become important. Hence the maximum charge  $Q_{\text{ext}}$  that can be loaded on a finite mass black hole generically deviates from  $M$ . Whether the limit  $Q_{\text{ext}}/M \rightarrow 1$  is approached from above or from below as  $M \rightarrow \infty$  is unambiguously determined within the low-energy effective theory of photons and gravitons. The answer depends on the leading higher derivative corrections to the Einstein-Maxwell theory, which are

$$\begin{aligned}
S_{\text{eff}} = \int d^4x \sqrt{-g} & \left[ \frac{1}{2} M_{\text{Pl}}^2 R - \frac{1}{4} M_{\text{Pl}}^2 F_{\mu\nu} F^{\mu\nu} \right. \\
& + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \\
& + c_4 R F_{\mu\nu} F^{\mu\nu} + c_5 R_{\mu\nu} F^\mu{}_\alpha F^{\nu\alpha} + c_6 R_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \\
& \left. + c_7 (F_{\mu\nu} F^{\mu\nu})^2 + c_8 F_{\mu\nu} F_{\alpha\beta} F^{\mu\alpha} F^{\nu\beta} + \text{higher derivatives} \right].
\end{aligned} \tag{1}$$

For large black holes, and in units where the electric field is  $F_{01} = \frac{Q}{4\sqrt{2}\pi M_{\text{Pl}}^2 r^2}$ , the maximum charge for a given mass  $M$  is modified to

$$\frac{Q_{\text{ext}}}{M} = 1 + \frac{128\pi^2 M_{\text{Pl}}^2}{5M^2} c_{\text{ext}} + O(M_{\text{Pl}}^4/M^4) \tag{2}$$

where as shown in [29] and recently revisited in [32] (whose conventions are adopted here)

$$c_{\text{ext}} = c_2 + 4c_3 + c_5 + c_6 + 4c_7 + 2c_8. \tag{3}$$

One might expect that in a consistent theory of quantum gravity the coefficients  $\{c_1, \dots, c_8\}$

are constrained such that  $c_{\text{ext}} > 0$ . This expectation is supported by the unitarity constraints on the Euler-Heisenberg action, which in particular imply  $4c_7 + 2c_8 > 0$  [34, 35, 36]. However despite earlier attempts the problem remained unsolved due to a challenge in generalizing the argument to the gravitational context. The forward scattering amplitude  $\mathcal{A}(s, t = 0)$ , which is the subject of study in [36], is singular in a gravitational theory because of the  $t$ -channel pole from the graviton exchange.

Recently two ways to get around this issue and to prove  $c_{\text{ext}} > 0$  have been proposed. The argument of [32] relies on the intriguing observation that the change in the entropy of maximally charged black holes is controlled by the same combination  $c_{\text{ext}}$ . The authors argue  $c_{\text{ext}}$  must be positive by employing a notion of monotonicity for the black hole entropy calculated in different theories. The proposal of [37] is instead to regulate the  $t$ -channel singularity of the forward amplitude by compactifying the theory to  $3d$  and using Eikonal resummation.

There are some subtleties in both arguments. In the first case they originate from our poor understanding of black hole micro-state counting. In the second case from the complications of gravitational scattering in  $3d$ . The goal here is to give a third, purely field-theoretic argument which is summarized below.

## 1.1 Summary

Our argument applies in the restricted context when below the strong gravity scale  $\Lambda$  there is a description in terms of a Quantum Field Theory (QFT) with a large number of degrees of freedom that are gapped at scales  $m_i$  hierarchically below the cutoff. To simplify the argument we assume the masses are all within a window near one scale  $m$  with  $\log(\Lambda/m) \gg 1$ . That is,

*(i) In the energy range  $m \ll E \ll \Lambda$ , there is an approximately scale invariant QFT with a large central charge  $C \gg 1$ .*

We denote this scale-invariant theory the UV CFT (see figure 1). The weak coupling to gravity can be thought of as an irrelevant deformation of the UV CFT with scale  $\Lambda$ . There are also relevant deformations with scale  $m$ . Moreover, we assume that at low-energies QFT is coupled (weakly) to the photon field:

*(ii) QFT has a nontrivial electromagnetic current  $J^\mu$  at  $E \sim m$ .*

Below  $m$  the only degrees of freedom are gravitons and photons described by the effective

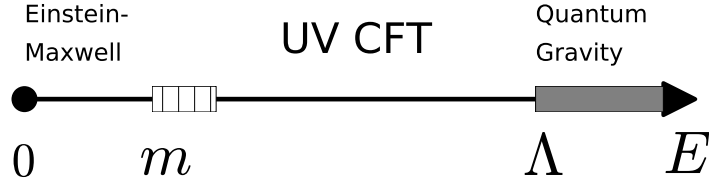


Figure 1: We assume an approximately scale-invariant QFT (called the UV CFT) below the quantum gravity scale  $\Lambda$ . UV CFT has a large number of degrees of freedom that are gapped at a hierarchically lower scale  $m$ . For  $E \ll \Lambda$  gravity is approximated as an external probe of the QFT.

action (1).

In this scenario, the running of the effective action for photons and gravitons between  $m$  and  $\Lambda$  is dominated by the QFT degrees of freedom, and hence gravitons can be treated as the external sources for the energy-momentum tensor. The same can be said about low-energy ( $E \sim m$ ) photons as external sources for the  $U(1)_{EM}$  current. The effective action can then be thought of as the generating functional for  $T^{\mu\nu}$  and  $J^\mu$  correlation functions of the QFT:

$$e^{iS_{\text{eff}}[A_\mu, g_{\mu\nu}]} = \int D\phi e^{iS[\phi, A_\mu, g_{\mu\nu}]} \quad (4)$$

where the path integral is over all QFT degrees of freedom denoted by  $\phi$ . A simple power-counting reveals that in four spacetime dimensions the running of the coefficients of the 4-derivative operators in (1) is not UV dominated. Hence we have a well-posed problem within the QFT.

Nevertheless there is a crucial input from gravity to this field theoretic argument. In a QFT one is free to add local counter-terms to  $S_{\text{eff}}$ , and therefore only the differences between the UV and IR coefficients are meaningful. In our case, the fact that gravity is dynamical and ultimately it becomes strong at  $\Lambda$  forbids arbitrarily large UV counter-terms. Once the running of the coefficients parametrically exceeds the allowed range of the UV values we can confidently talk about the absolute values of the IR coefficients.

To illustrate this point let's momentarily ignore the Euler-Heisenberg terms (those schematically of the form  $RF^2$  and  $F^4$ ) in (1). The correction to the extremality condition is then controlled by  $c_{\text{ext}} \rightarrow c_2 + 4c_3$ . In section 2 we will focus on this case to give a detailed formulation of the setup. In particular, we will discuss possible non-minimal couplings to gravity and explain why our derivation is robust against field redefinitions as long as they do not introduce spuriously large irrelevant deformations to the UV CFT. We will see that

there is a more suitable basis for the curvature-squared corrections

$$\int d^4x \sqrt{-g} \left[ c_{R^2} R^2 + c_{W^2} W_{\mu\nu\alpha\beta} W^{\mu\nu\alpha\beta} + c_{\text{GB}} \mathcal{L}_{\text{GB}} \right] \quad (5)$$

where  $W_{\mu\nu\alpha\beta}$  is the Weyl tensor and the Gauss-Bonnet term is defined as

$$\mathcal{L}_{\text{GB}} = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2. \quad (6)$$

In this basis  $c_2 + 4c_3 = 2c_{W^2}$ .

We will derive a sum-rule for the running of  $c_{W^2}$  in terms of the norm-square of matrix elements of  $T_{\mu\nu}$ . This coefficient has a parametrically large running during the CFT phase due to the Weyl anomaly [38]. Hence up to small corrections (from UV ambiguities and Renormalization Group (RG) flow near  $E \sim m$ ), we obtain

$$c_{\text{ext}} \simeq 2C \log \frac{\Lambda}{m} + \text{Contribution from } RF^2 \text{ and } F^4. \quad (7)$$

The central charge  $C$  of the UV CFT determines the normalization of the  $T^{\mu\nu}$  two-point function. In momentum space this translates into a non-analytic piece:

$$\varepsilon_{\mu\nu} \varepsilon_{\alpha\beta} \langle T^{\mu\nu}(p) T^{\alpha\beta}(-p) \rangle = 2C p^4 \log(p^2/\Lambda^2), \quad (8)$$

where  $p^\mu$  is a time-like four-momentum with norm  $p^2 = p_\mu p^\mu$  and  $\varepsilon^{\mu\nu}$  is a normalized, real, transverse-traceless polarization tensor. By unitarity  $C > 0$  and it is large by assumption (i). Hence  $c_{\text{ext}} \geq 0$  and  $Q_{\text{ext}}/M - 1 > 0$  if Euler-Heisenberg terms are ignored.

In section 3 we derive QFT sum-rules for the running of the Euler-Heisenberg interactions,  $\{c_4, \dots, c_8\}$ . We will argue that the sums, which by our assumption (ii) are nontrivial, are infra-red (IR) dominated. Therefore, having fixed the UV scale  $\Lambda$ , once  $m \ll \Lambda$  the IR coefficients would grow parametrically larger than the UV values.

Focusing on the purely electromagnetic sector, we can choose the alternative basis

$$\int d^4x \sqrt{-g} \left[ c_{F^4} (F_{\mu\nu} F^{\mu\nu})^2 + \tilde{c}_{F^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right], \quad (9)$$

where  $\tilde{F}^{\mu\nu} = \frac{1}{4} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ . In this basis one obtains positive-definite sums for both  $c_{F^4}$  and  $\tilde{c}_{F^4}$ . In particular, the extremality condition depends on  $4c_7 + 2c_8 = 4c_{F^4} > 0$ .

Hence the fully electromagnetic and the fully gravitational contribution to  $c_{\text{ext}}$  are positive. We will then use the IR dominance of the sum-rules for  $F^4$  and  $RF^2$  interactions, to truncate the sums at some  $\Lambda'$  such that  $m \ll \Lambda' \ll \Lambda$ , but  $\log(\Lambda'/m) \sim 1$ . From the

truncated sums our final result follows:

$$c_{\text{ext}} \geq 2C \log \frac{\Lambda}{m} - \sqrt{112c_{F^4}C \log(\Lambda'/m)} + 4c_{F^4} > 0, \quad \text{if } \log \frac{\Lambda}{m} \gg 1. \quad (10)$$

Further comments will follow in section 4.

## 2 Curvature Corrections

Our goal in this section is to relate the coefficients of the low-energy gravitational effective action  $c_1, c_2, c_3$  to the QFT data by treating the metric as an external source. In this approximation the object of interest  $S_{\text{eff}}[g_{\mu\nu}]$  is the generating functional for the correlation functions of the QFT stress-energy tensor  $T^{\mu\nu}$ . In order to determine the running of the curvature-squared terms in  $S_{\text{eff}}$ , we first have to explore the space of possible deformations of the UV CFT through non-minimal coupling to the metric (section 2.1). This will be used to show the robustness of our final result (7), for instance under metric field redefinition. In section 2.2 we will introduce graviton “pseudo-amplitudes” as the momentum-space correlation functions of  $T^{\mu\nu}$  and in section 2.3 write down the dispersion relations. In section 2.4 we use the optical theorem to derive the sum-rules for the coefficients  $c_{R^2}$  and  $c_{W^2}$  in the basis (5). Finally the sum for  $c_{W^2}$  is evaluated in section 2.5 to derive the purely gravitational contribution to the extremality bound.

### 2.1 Non-minimal Couplings

Apart from the minimal coupling (replacing  $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ ) general covariance allows for coupling of QFT operators to the curvature tensor and its derivatives. By including the trivial case where the QFT operator in question is the identity  $\mathbf{1}$ , the space of all these “non-minimal couplings” covers also the local counter-terms that can be added to  $S_{\text{eff}}[g_{\mu\nu}]$  in the UV CFT. As usual there is at most a finite number of relevant and marginal deformations and an infinite set of irrelevant ones. Below we will discuss these deformations, following [39].

The most relevant term in this list is the cosmological constant which has to be tuned to zero. The next one is the Einstein-Hilbert term, which can be thought of as coupling scalar curvature  $R$  to the unit operator  $\mathbf{1}$ . Its coefficient, the Planck scale  $M_{\text{Pl}}^2$ , is a UV sensitive quantity with little significance if we were not dealing with a gravitational theory. Instead, now the Planck scale plays the important role of fixing the UV cutoff  $\Lambda$  as the scale at which the QFT contribution to the graviton correlation functions becomes significant. In particular, the graviton 2-point function receives a local piece from the Einstein-Hilbert

term and a non-local piece (8) from coupling to matter  $T^{\mu\nu}$ . Schematically

$$\varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}\frac{\delta^2 S_{\text{eff}}}{\delta h_{\mu\nu}(p)\delta h_{\alpha\beta}(-p)}\sim M_{\text{Pl}}^2 p^2 + N p^4, \quad (11)$$

where  $N \sim C$  is the characteristic number of degrees of freedom. The energy at which the two terms become comparable determines the cutoff

$$M_{\text{Pl}}^2 \sim N \Lambda^2. \quad (12)$$

The next subleading couplings of metric to the CFT unit operator are the exactly marginal curvature-squared terms listed in (1). Because of large logarithms their coefficients are expected to be very different in the UV with respect to the low-energy theory. We assume that once the RG scale  $m$  is taken much below  $\Lambda$  the UV coefficients become independent of  $m$ . Hence, consistent with the above definition of the UV cutoff, the UV coefficients of the curvature-squared corrections are taken to be at most

$$c_{1,2,3}^{UV} \sim N. \quad (13)$$

So far we only discussed the local terms that one can add directly to the effective action  $S_{\text{eff}}[g_{\mu\nu}]$ . Next consider coupling to nontrivial CFT operators. The only way to have a relevant (marginal) deformation among those is if the CFT contains a singlet operator  $\mathcal{O}$  with dimension  $\Delta_{\mathcal{O}} < 2$  ( $\Delta_{\mathcal{O}} = 2$ ). In a perturbative setup this can be a real scalar field  $\mathcal{O}_1 = \phi$ , with  $\Delta_1 \simeq 1$ , or  $\mathcal{O}_2 = \phi^2$ , with  $\Delta_2 \simeq 2$ . If such an operator  $\mathcal{O}$  exists there is no reason to exclude the coupling

$$S_R[\mathcal{O}] = \int d^4x \sqrt{-g} \mu^{2-\Delta_{\mathcal{O}}} R \mathcal{O}, \quad (14)$$

for some mass scale  $\mu$ . In fact there is a particular case where this coupling is dictated by symmetry. If the above-mentioned field  $\phi$  is the Goldstone boson of a spontaneously broken global symmetry, its shift-symmetry forbids conformal coupling to the metric. Hence, one can think of the minimally coupled scalar  $\phi$  as a conformally coupled one plus an unimprovement term  $S_R[\phi^2]$  [39]. Depending on the relative size of  $\mu$  and the RG scale  $m$ , this deformation can have a significant effect on the IR coefficients  $c_i$ . Nevertheless we will see that the particular combination  $c_2 + 4c_3$ , which enters  $Q_{\text{ext}}/M$ , is unaffected by these couplings.<sup>1</sup>

Unitarity constraints on the dimension of higher spin operators ensure that all other

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<sup>1</sup>We mention in anticipation that the potential effect on the Euler-Heisenberg term  $c_4$  in (1) is also irrelevant for the extremality condition.

couplings to the curvature tensor are irrelevant. We assume all these irrelevant couplings are suppressed at least by  $\Lambda$  (since we are considering  $N \gg 1$ , some couplings will also include suppression by  $N$  to ensure that the deformation is small when  $E \ll \Lambda$ ). An example of such an irrelevant coupling is

$$S_{R_{\mu\nu}}[T] = \int d^4x \sqrt{-g} \frac{1}{2\Lambda^2} R_{\mu\nu} T^{\mu\nu}, \quad (15)$$

which would be generated under a redefinition of the metric

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \frac{1}{\Lambda^2} R_{\mu\nu}. \quad (16)$$

Applying the same field-redefinition in the low-energy theory changes the  $c_i$  coefficients in (1). However the above suppression ensures that

$$\Delta c_{1,2,3}^{IR} \sim N. \quad (17)$$

Equations (13) and (17) ensure that the ambiguity in the IR values  $c_{1,2,3}^{IR}$  are at most  $O(N) = O(C)$  which is parametrically smaller than our result (7).

## 2.2 Graviton Pseudo-Amplitudes

The derivation of (7) is based on dispersion relations written for the connected time-ordered correlation functions of  $T^{\mu\nu}$ . In our setup the real-space time-ordered correlation functions at non-coincident points are related to the gravitational effective action as

$$\frac{i^n}{2^n} \langle T[T^{\mu_1\nu_1}(x_1) \cdots T^{\mu_n\nu_n}(x_n)] \rangle_c = \frac{\delta^n}{\delta g_{\mu_1\nu_1}(x_1) \cdots \delta g_{\mu_n\nu_n}(x_n)} iS_{\text{eff}}[g_{\mu\nu}] \Big|_{g_{\mu\nu}=\eta_{\mu\nu}}. \quad (18)$$

Going to the momentum space and contracting the  $i$ th stress-energy tensor by the polarization vector  $\varepsilon_{(\lambda_i)}^{\mu\nu}$ , we obtain graviton “pseudo-amplitudes” [40]:

$$(2\pi)^4 \delta^4\left(\sum_{i=1}^n k_i\right) i\mathcal{A}(k_1, \dots, k_n; \lambda_1 \cdots \lambda_n) = \frac{\delta^n}{\delta h(k_1, \lambda_1) \cdots \delta h(k_n, \lambda_n)} iS_{\text{eff}}[g_{\mu\nu}] \Big|_{g_{\mu\nu}=\eta_{\mu\nu}}, \quad (19)$$

where we have Fourier-transformed  $h_{\mu\nu}(x) \equiv g_{\mu\nu}(x) - \eta_{\mu\nu}$  and expanded in terms of a complete set of polarization vectors

$$h_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \sum_{\lambda} h(k, \lambda) \varepsilon_{\mu\nu}^{(\lambda)} e^{ik \cdot x}. \quad (20)$$



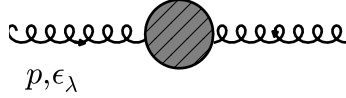


Figure 2: We write a dispersion relation for the off-shell graviton pseudo-amplitude with momentum  $p = (\sqrt{s}, \vec{0})$ . As explained in the text pseudo-amplitudes have no internal gravitons and hence are the momentum space time-ordered correlation functions of the QFT stress tensor.

Pseudo-amplitudes are amputated amplitudes with only external gravitons. Diagrammatically they are given by a sum just over the tree-level contact diagrams, with vertices derived from  $S_{\text{eff}}$ .

Unlike on-shell scattering amplitudes, pseudo-amplitudes are not uniquely defined. They depend on how graviton field  $h_{\mu\nu}$  is defined at nonlinear order. For instance, the same  $n$ -point amplitude defined in terms of  $h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$  would generically be different. This reflects the ambiguity of the real-space correlation functions (18) at coincident points. However we are only concerned with the 2-graviton amplitude, which is unaffected by these nonlinear transformations since there is no tadpole term in  $S_{\text{eff}}$ .

On the other hand field redefinitions that start at linear order contain derivatives and introduce irrelevant couplings in the UV CFT, as in (16). The requirement that irrelevant couplings are suppressed by  $\Lambda$  ensures that the effect of these redefinitions on  $c_{\text{ext}}$  is parametrically suppressed, going at most as  $N$  with no logarithmic enhancement.

## 2.3 Dispersion Relations

Our sum-rules follow from the standard dispersion relations, as discussed in [41], but written for  $\langle T[T^{\mu\nu}(x)T^{\alpha\beta}(0)] \rangle$ . That is to say, we are working with the forward 2-graviton pseudo-amplitude  $\mathcal{A}_{h \rightarrow h}$ , figure 2. The minor deviations from [41] are to simplify the generalization to include the 3- and 4- point functions in the next section.

As the external momentum we choose  $p^\mu = (\sqrt{s}, 0, 0, 0)$ , and write the dispersion relation in terms of  $s$ . The pseudo-amplitude  $\mathcal{A}_{h \rightarrow h}(s, \lambda)$  is analytic except for a discontinuity along the positive real  $s$ . Moreover  $\mathcal{A}$  is analytic below the gap  $s < m^2$ , where  $S_{\text{eff}}$  admits a derivative expansion in terms of local terms (1). The analyticity near  $s = 0$  will break down once the photon field is included as a dynamical QFT degree of freedom. Loops of photons

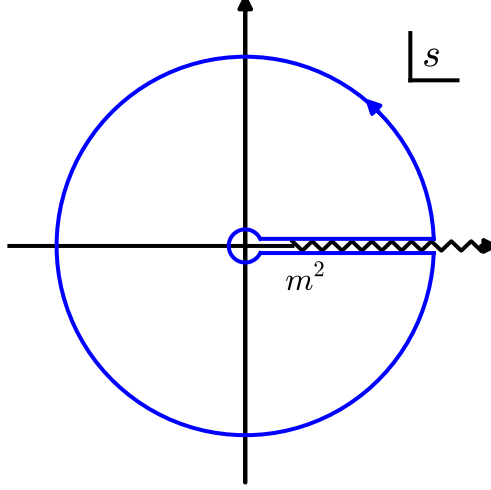


Figure 3: The complex  $s$  plane contour. The discontinuity of the amplitude is related to the exchange of QFT degrees of freedom and starts from  $s \sim m^2$ .

make the running, and hence the discontinuity, extend all the way to  $s = 0$ . This effect is small since it is relatively suppressed by  $N$  and goes in the same direction as other QFT contributions. So to keep things simple we will neglect it.

We are interested in the 4-derivative operators, which contribute at order  $s^2$ . A twice subtracted dispersion relation extracts this contribution (see figure 3):

$$i\pi\mathcal{A}_{h\rightarrow h}''(s=0, \lambda) = \oint_{|s|=\Lambda^2} \frac{ds}{s^3} A_{h\rightarrow h}(s, \lambda) + \int_{m^2}^{\Lambda^2} \frac{ds}{s^3} \text{Disc}\mathcal{A}_{h\rightarrow h}(s, \lambda), \quad (21)$$

where prime denotes derivative with respect to  $s = -p^2$ .

Let us first discuss the choice of the polarization tensor. By diff-invariance  $\varepsilon_{\mu\nu}^{(\lambda)}$  can be taken to be fully spatial. The independent choices are then a pure trace  $\varepsilon_{ij}^{(0)} \propto \delta_{ij}$  corresponding to zero angular momentum  $l = 0$ , and the five symmetric-traceless tensors corresponding to  $l = 2$ . All polarization tensors are chosen real. Working in the basis (5) of curvature-squared operators, which is related to the original basis (1) via

$$\begin{aligned} c_1 &= c_{R^2} + c_{\text{GB}} + \frac{1}{3}c_{W^2} \\ c_2 &= -4c_{\text{GB}} - 2c_{W^2} \\ c_3 &= c_{\text{GB}} + c_{W^2}, \end{aligned} \quad (22)$$

the  $l = 0$  polarization singles out  $R^2$ , and  $l = 2$  polarization  $W^2$ :

$$\mathcal{A}_{h \rightarrow h}''(s = 0, \lambda) = \varepsilon_{\mu\nu}^{(\lambda)} \left[ 12c_{R^2} P_0^{\mu\nu\alpha\beta} + 2c_{W^2} P_2^{\mu\nu\alpha\beta} \right] \varepsilon_{\alpha\beta}^{(\lambda)}, \quad (23)$$

where the  $l = 0$  and  $l = 2$  projectors are defined respectively as

$$\begin{aligned} P_0^{\mu\nu\alpha\beta} &= \frac{1}{3} \hat{\eta}^{\mu\nu} \hat{\eta}^{\alpha\beta}, \\ P_2^{\mu\nu\alpha\beta} &= \frac{1}{2} \left[ \hat{\eta}^{\mu\alpha} \hat{\eta}^{\nu\beta} + \hat{\eta}^{\mu\beta} \hat{\eta}^{\nu\alpha} - \frac{2}{3} \hat{\eta}^{\mu\nu} \hat{\eta}^{\alpha\beta} \right], \end{aligned} \quad (24)$$

and  $\hat{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$ . For our choice of  $p^\mu$  these are all purely spatial matrices. The Gauss-Bonnet term is a total derivative in  $4d$  and hence does not contribute to the momentum space pseudo-amplitudes.

## 2.4 Optical Theorem and The Sum-Rules

Next we use the “optical theorem” to relate  $\text{Disc}\mathcal{A}$  to the norm-square of matrix elements of  $T^{\mu\nu}$ . The discontinuity of  $\mathcal{A}_{h \rightarrow h}$  across the half line  $s > m^2$  coincides with twice the imaginary part, and can be obtained from the following identity for the time-ordered and anti-time-ordered ( $\bar{T}$ ) correlation functions [42, 40]:

$$\sum_{k=0}^n (-1)^k \sum_{\sigma \in \Pi(k, n-k)} \bar{T}[\mathcal{O}(x_{\sigma_1}) \cdots \mathcal{O}(x_{\sigma_k})] T[\mathcal{O}(x_{\sigma_k}) \cdots \mathcal{O}(x_{\sigma_n})] = 0 \quad (25)$$

where  $\Pi(k, n-k)$  is the set of all partitions  $\sigma$  of  $\{1, 2, \dots, n\}$  into two sets of size  $k$  and  $n-k$ . In the simple case of  $n = 2$ , this identity relates the discontinuity of  $\mathcal{A}_{h \rightarrow h}(s, \lambda)$  to the momentum-space Whiteman function

$$\text{Disc}\mathcal{A}(s > m^2, \lambda) = \frac{i}{4} \varepsilon_{\mu\nu}^{(\lambda)} \varepsilon_{\alpha\beta}^{(\lambda)} \langle 0 | T^{\mu\nu}(-p) T^{\alpha\beta}(p) | 0 \rangle_{p=(\sqrt{s}, \vec{0})}, \quad (26)$$

where on the r.h.s. we kept only the one ordering that is nonzero for the physical momentum  $p^0 > 0$ . The sum-rule (and also the fact that the opposite ordering is zero) follow from insertion of a complete set of QFT states between the two operators

$$\mathbb{I} = \sum_n \int \frac{d^3 \vec{p}_n}{(2\pi)^3 2E_n} \sum_{l, m} |n, l, m, \vec{p}_n\rangle \langle n, l, m, \vec{p}_n| \quad (27)$$

where  $l, m$  parametrize the spin, and  $\vec{p}_n$  the total 3-momentum. The rest-mass  $M_n$  and

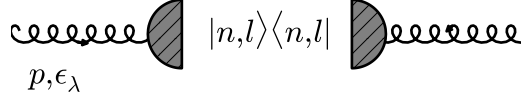


Figure 4: The discontinuity of graviton pseudo-amplitude can be calculated by insertion of a complete set of QFT states. Only states with total angular momentum  $l = 0, 2$  have a nonzero matrix element.

other quantum numbers are implicit in  $n$ . Only QFT states with angular momentum  $l = 0$  and  $l = 2$  can have nonzero matrix elements with the “off-shell graviton”. Introducing the notation

$$|T_\lambda(p)\rangle \equiv \varepsilon_{\mu\nu}^{(\lambda)} \int d^4x e^{ip \cdot x} T^{\mu\nu}(x) |0\rangle, \quad (28)$$

for  $l = 0$  we write

$$\langle T_\lambda(p) | n, 0, \vec{p}_n \rangle = \frac{1}{\sqrt{3}} \alpha_{n,0} \text{Tr}[\varepsilon_{(\lambda)}] (2\pi)^4 \delta^4(p_n^\mu - p^\mu) \quad (29)$$

where  $p_n^0 = \sqrt{|\vec{p}_n|^2 + M_n^2}$  and for  $l = 2$

$$\langle T_\lambda(p) | n, 2, m, \vec{p}_n \rangle = \alpha_{n,2} \text{Tr}[\varepsilon^{(\lambda)} \cdot \varepsilon^{(2,m)}] (2\pi)^4 \delta^4(p_n^\mu - p^\mu). \quad (30)$$

Using the completeness relation for polarization vectors (for  $l = 2$  in particular)

$$\sum_m \varepsilon_{(2,m)}^{\mu\nu} \varepsilon_{(2,m)}^{\alpha\beta} = P_2^{\mu\nu\alpha\beta} \quad (31)$$

and comparing with (23) gives

$$\begin{aligned} c_{R^2} &= c_{R^2}^{UV} + \sum_{n,l=0} \frac{|\alpha_{n,0}|^2}{24M_n^6} \\ c_{W^2} &= c_{W^2}^{UV} + \sum_{n,l=2} \frac{|\alpha_{n,2}|^2}{4M_n^6}, \end{aligned} \quad (32)$$

where we denoted the contour integral at  $|s| = \Lambda^2$  by the superscript UV, preserving  $c_{R^2}, c_{W^2}$  for the IR parameters. This positivity of the running of  $c_{R^2}$  and  $c_{W^2}$  toward IR is an old result [43]. As discussed above, we assume the UV amplitude becomes independent of  $m$  once  $m \ll \Lambda$ . Hence  $c^{UV}$  are expected to be at most of order  $N$  as estimated in (13).

A potential subtlety is that the stress-energy tensor  $T^{\mu\nu}$  appearing in the above formulas is the QFT operator that is sourced by  $h_{\mu\nu}$ . Hence its correlation functions depend on the non-minimal coupling to the curvature invariants. As seen, these comprise not only local terms added in the UV to  $S_{\text{eff}}[g_{\mu\nu}]$ , which have no effect on the  $\text{Disc}\mathcal{A}$ , but also coupling to local QFT operators which do have an effect. Most of these are also irrelevant couplings whose contribution to the dispersion relation is comparable to  $c_{1,2,3}^{UV}$ .

The exceptions to this rule are the relevant/marginal deformations (14). In particular, their presence would lead to a non-vanishing  $\text{Disc}\mathcal{A}$  for the trace of stress-energy tensor even in the conformal range  $m \ll E \ll \Lambda$ . A familiar example is a minimally coupled scalar field. As discussed below (14) this theory can be thought of as a conformally coupled scalar field plus  $S_R[\xi\phi^2]$ , with  $\xi = 1/12$ . This would imply a parametrically large IR coefficients  $c_{R^2}$ :

$$c_{R^2} = \frac{\xi^2}{4\pi^2} N_s \log \frac{\Lambda}{m} + O(N), \quad (33)$$

where  $N_s$  is the number of free minimally coupled scalar fields gapped at  $m$ . Hence depending on the composition of the UV CFT and the relevant/marginal deformations (14), the low-energy coefficient  $c_{R^2}$  may or may not be unambiguously determined.

## 2.5 Conformal Anomaly and Black Hole Extremality Bound

Luckily the extremality condition  $Q_{\text{ext}}/M$  depends on the combination  $c_2 + 4c_3$  (c.f. (3)) which is independent of both  $c_{\text{GB}}$  and  $c_{R^2}$ . The relations (22) imply

$$c_2 + 4c_3 = 2c_{W^2}. \quad (34)$$

The deformations (14) have no effect on the  $\mathcal{A}_{h \rightarrow h}(s, l=2)$  and hence up to a parametrically small UV ambiguity and a parametrically small contribution from the RG at  $E \sim m$ , the IR value of  $c_{W^2}$  is universally fixed in terms of the CFT central charge  $C$  as follows.

First we relate the discontinuity of the stress tensor 2-point function to the  $C$  anomaly, as pointed out in (8). Consider the CFT 2-point correlation function of the stress tensor in momentum space and with a normalized traceless polarization  $\text{Tr}[\varepsilon^{(\lambda)} \cdot \varepsilon^{(\lambda)}] = 1$ . By dimension counting

$$\mathcal{A}_{h \rightarrow h}^{CFT}(s, l=2) = s^2 f(s/\Lambda^2). \quad (35)$$

The discontinuity of the function  $f$  is an intrinsic data of the CFT (see [42, 40] for other applications of the same idea). It can be determined by performing an infinitesimal rigid rescaling with parameter  $\epsilon$ . The change in the CFT effective action is

$$\delta S_{\text{eff}}[g_{\mu\nu}] = \epsilon \int d^4x \sqrt{-g} [C W_{\mu\nu\alpha\beta} W^{\mu\nu\alpha\beta} + A \mathcal{L}_{\text{GB}}] \quad (36)$$

where  $C$  and  $A$  are the anomaly coefficients. The Gauss-Bonnet term is a total derivative and does not contribute to the momentum space amplitudes, while the  $C$ -anomaly term gives (c.f. (23))

$$\delta \mathcal{A}_{h \rightarrow h}^{CFT}(s, l = 2) = \epsilon C s^2 = \epsilon (4 - 2s \partial_s) \mathcal{A}_{h \rightarrow h}^{CFT}(s, l = 2) \quad (37)$$

where in the last step we wrote the rescaling operator. This fixes

$$f(s/\Lambda^2) = -\frac{1}{2} C \log(-s/\Lambda^2), \quad (38)$$

up to analytic terms. Substituting the discontinuity in (21) gives

$$c_{W^2} = C \log \frac{\Lambda}{m}, \quad (39)$$

with parametrically small corrections. We conclude

$$c_{\text{ext}} \simeq 2C \log \frac{\Lambda}{m} + \text{Contribution from } RF^2 \text{ and } F^4. \quad (40)$$

### 3 Including Euler-Heisenberg Interactions

We are assuming that there are electromagnetically charged states in the QFT. They lead to the running of the coefficients of the Euler-Heisenberg interactions, those schematically of the form  $RF^2$  and  $F^4$ . The goal here is to quantify this running. We will start in section 3.1 by discussing the fate of  $U(1)_{EM}$  in the UV. In section 3.2 we introduce the pseudo-amplitudes for on-shell photons as quantities that are well-defined regardless of the fate of  $U(1)_{EM}$ . In section 3.3 dispersion relations are derived. In addition to the 2-point amplitude  $\mathcal{A}_{h \rightarrow h}$ , the “decay” amplitude  $\mathcal{A}_{h \rightarrow AA}$ , its time-reversal  $\mathcal{A}_{AA \rightarrow h}$  and the photon-photon forward amplitude  $\mathcal{A}_{AA \rightarrow AA}$  are studied. In section 3.4 optical theorem is used to derive sum-rules for  $\{c_4, \dots, c_8\}$ . Finally in section 3.5 the sum-rules are combined to prove the positivity of  $c_{\text{ext}}$ .

### 3.1 $U(1)_{EM}$

It is useful to introduce the dimensionfull photon field  $B_\mu$  as the source for the conserved  $U(1)_{EM}$  current  $J^\mu$  of the QFT. At low energies electromagnetism is weak and  $B_\mu$  can be treated, just like  $g_{\mu\nu}$ , as an auxiliary field. Its dimension is  $\simeq 1$ , since it sources a conserved current which has dimension 3. The lowest order gauge-invariant action we can write for  $B_\mu$  is

$$S_{\text{Maxwell}} = -\frac{1}{4g^2} \int d^4x \sqrt{-g} B_{\mu\nu} B^{\mu\nu} \quad (41)$$

where  $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$  is the field strength. The IR value of the coupling  $g$  determines the normalization of the dimensionless  $A_\mu$  field in (1)

$$B_\mu = g M_{\text{Pl}} A_\mu. \quad (42)$$

A QFT state that transforms under an integer  $q$  representation of  $U(1)$ , couples to  $A_\mu$  with strength  $qgM_{\text{Pl}}$ . This is the electric charge of the state in Planck unites. The assumption (ii) is that such states exist in the QFT.

The next-to-leading terms are the 4-derivative Euler-Heisenberg interactions. These are irrelevant operators,

$$[RB_{\mu\nu}^2] = 6, \quad [B_{\mu\nu}^4] = 8. \quad (43)$$

If one could assume that the weakly gauged  $U(1)_{EM}$  exists all the way to  $E \sim \Lambda$ , the rest of the analysis would closely parallel that of the last section. The irrelevant operators could be added to the UV CFT but with appropriate suppression. For instance, assuming that there are  $O(N)$  charged degrees of freedom in the QFT and also that  $U(1)_{EM}$  emerges (as discussed in [24, 25]) at  $E \sim \Lambda$ , we have  $1/g_{UV}^2 \sim M_{\text{Pl}}^2/\Lambda^2 \sim N$ . Then it would be natural to add

$$\int d^4x \sqrt{-g} \frac{1}{g_{UV}^2 \Lambda^4} (B_{\mu\nu} B^{\mu\nu})^2, \quad (44)$$

which after normalizing gives

$$c_7^{UV} \sim \frac{g^4 M_{\text{Pl}}^4}{g_{UV}^2 \Lambda^4} \sim N \frac{g^4}{g_{UV}^4}. \quad (45)$$

This would be parametrically smaller than the contribution of the sum-rules to the IR values.

However having electromagnetism weakly coupled throughout the whole range of scales is not the most generic scenario to consider. The  $U(1)_{EM}$  might be emergent at low energies. For instance it could combine into a bigger gauge group within QFT, as in the Standard Model. In this case it is inconsistent to treat photons as external sources all the way to the UV. A priori, they have to be included in the unitarity sums, and as internal states.

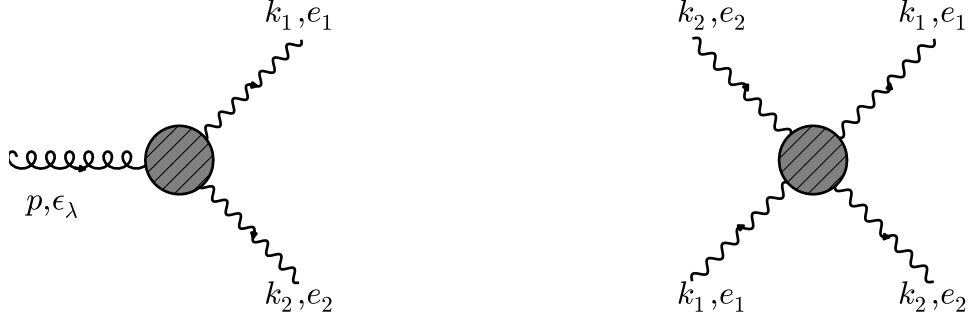


Figure 5: The pseudo-amplitude for an off-shell graviton decaying into two on-shell photons (plus its time-reversal) and forward 2-to-2 amplitude for photons are used to derive sum-rules for the coefficients of  $RF^2$  and  $F^4$  interactions.

### 3.2 Photon and Graviton pseudo-amplitudes

Whether or not a weakly coupled photon field exists at all scales, the on-shell photon amplitudes are well-defined at all energies since photons are the asymptotic states. In a Lorentz invariant theory the scattering states made of well-separated highly boosted photons is constructed within the low-energy effective theory. We will therefore write dispersion relations that involve on-shell photons. Because gravitons are always treated as external sources, all momentum-space amplitudes discussed here are “pseudo-amplitudes”, though for brevity we often drop the prefix.

Amplitudes with on-shell photons are related via LSZ reduction to the time-ordered products of the  $U(1)_{EM}$  current  $J^\mu$  (see e.g. [41]). Hence we will be dealing with time-ordered correlation functions that include  $J^\mu$ , in addition to  $T^{\mu\nu}$ .<sup>2</sup>

In particular, in addition to the 2-point graviton amplitude at off-shell momentum  $p^\mu = (\sqrt{s}, \vec{0})$  here we will be using the following amplitudes: (a) The decay of the same off-shell graviton into two on-shell photons (i.e. with  $k_1^2 = k_2^2 = 0$  and transverse polarizations  $e_1^\mu$  and  $e_2^\mu$ ) plus its time-reversal, and (b) the 2-to-2 amplitude of on-shell photons (see figure 5). The latter is a true scattering amplitude of the QFT if gravity is fully decoupled.

<sup>2</sup>If photons were treated as external sources for the QFT, then every non-coincident insertion of the current operator in a time-ordered correlation function would be given by an extra derivative of the QFT generating functional for  $J^\mu$  and  $T^{\mu\nu}$  with respect to  $A_\mu$ :

$$\langle T[\dots J^\mu(x)] \rangle \rightarrow \dots \frac{-i}{gM_{Pl}} \frac{\delta}{\delta A_\mu(x)} iS_{\text{eff}} \Big|_{A_\mu=0, g_{\mu\nu}=\eta_{\mu\nu}}. \quad (46)$$

Transforming to the momentum space and contracting by polarization vectors gives amputated momentum space amplitudes with only external photons and gravitons. As discussed above, while this is a good approximation in the infra-red, it is not necessarily justified at higher energies.



These amplitudes are given by

$$\begin{aligned}
\mathcal{A}_{h \rightarrow h}(s, \lambda) &= \frac{i}{4} \varepsilon_{\lambda}^{\mu\nu} \varepsilon_{\lambda}^{\alpha\beta} \int d^4x e^{ip \cdot x} \langle T[T_{\mu\nu}(x) T_{\alpha\beta}(0)] \rangle, \\
\mathcal{A}_{h \rightarrow AA}(s, \lambda, e_1, e_2) &= -\frac{g^2 M_{\text{Pl}}^2}{2} \varepsilon_{\lambda}^{\mu\nu} e_1^{\alpha} e_2^{\beta} \int d^4x d^4y e^{ip \cdot x - ik_1 \cdot y} \langle T[T_{\mu\nu}(x) J_{\alpha}(y) J_{\beta}(0)] \rangle, \\
\mathcal{A}_{AA \rightarrow AA}(s, t, \{e_i\}) &= -ig^4 M_{\text{Pl}}^4 e_1^{\mu} e_2^{\nu} e_3^{\alpha} e_4^{\beta} \\
&\quad \int d^4x d^4y d^4z e^{ik_1 \cdot x + ik_2 \cdot y + ik_3 \cdot z} \langle T[J_{\mu}(x) J_{\nu}(y) J_{\alpha}(z) J_{\beta}(0)] \rangle,
\end{aligned} \tag{47}$$

where the Mandelstam invariants for the 2-to-2 process are defined as

$$s = -(k_1 + k_2)^2 = -p^2, \quad t = -(k_1 + k_3)^2, \quad u = -(k_1 + k_4)^2. \tag{48}$$

We are interested in the forward limit:  $k_3 = -k_1$  (equivalently  $t = 0$ ),  $e_3 = e_1$ , and  $e_4 = e_2$ . However, before moving on to the derivation of dispersion relations, we will discuss a few potential subtleties.

Firstly we expect no IR singularities despite having massless photons. This is essentially because all other QFT degrees of freedom are gapped and the effective theory of photons is infra-red free. One could still worry about potential singularity of the forward limit  $t \rightarrow 0$  at fixed  $s$ . Generally this could arise from the  $t$ -channel exchange of almost on-shell massless particles. Therefore in our case the amplitude for this process would factorize into two on-shell amplitudes  $\mathcal{A}_{13,t}$  for the photons 1 and 3 going into the  $t$ -channel exchanged photons and  $\mathcal{A}_{24,t}$  for the same  $t$ -channel exchanged photons going to photons 2 and 4. But since in the  $t \rightarrow 0$  limit there is no nonzero invariant momentum in  $\mathcal{A}_{13,t}$  and  $\mathcal{A}_{24,t}$ , gauge and Lorentz invariance would force them to vanish.

The amplitudes have to be sufficiently well-behaved also in the UV for the dispersion relations to be useful in determining the IR coefficients. In particular, apart from the analytic terms that come from local counter-terms and hence are assumed to be suppressed by powers of  $\Lambda$ , we need the forward amplitude to grow slower than  $s^2$  at large  $s$ . Namely the Regge intercept must be  $l_0 < 2$ . We do not have a general proof for this, although it does seem to be plausible based on the argument given in the appendix. Ultimately, this has to be regarded as a technical assumption for the final positivity bound on  $c_{\text{ext}}$  to hold.

Another potential subtlety comes from the field redefinitions. Generically pseudo-amplitudes are not invariant under field redefinitions. However in our setup the ambiguities either vanish or they are parametrically suppressed. One class of field redefinitions correspond to adding

nonlinear terms with more powers of  $h_{\mu\nu}$ , as for instance is the case in the relation between  $A_\mu$  and  $A^\mu$ . In our case this would naively affect  $\mathcal{A}_{h\rightarrow AA}$ , when inserted in the Maxwell term, since it generates some cubic terms. However these cubic terms vanish because photons are on-shell.

One could also consider metric field redefinitions which necessarily involve derivatives or more  $B_\mu$  fields and hence they introduce irrelevant non-minimal couplings in the UV CFT, e.g.

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \frac{1}{\mu^4} B_{\mu\rho} B_\nu{}^\rho. \quad (49)$$

Choosing  $\mu \sim \Lambda$  ensures that the ambiguity they introduce in  $c_{\text{ext}}$  is parametrically suppressed.

### 3.3 Dispersion Relations

The combination  $\mathcal{A}_{h\rightarrow AA} + \mathcal{A}_{AA\rightarrow h}$  and the forward amplitude  $\mathcal{A}_{AA\rightarrow AA}|_{t=0}$  are analytic functions of  $s$  except for discontinuities along the real line. Once massless photons are treated as dynamical the discontinuity extends all the way to  $s = 0$ . However, we are working under the assumption that the number of QFT degrees of freedom is large and electromagnetism is weak at low-energies  $E \sim m$ . Hence it is justified to neglect loops of photons below the gap  $m$ , and for convenience we will do so. Then the discontinuity of the decay amplitude is along  $s > m^2$ . The discontinuity of the forward amplitude is along  $s > m^2$  and  $s < -m^2$  where physical states are exchanged respectively in the  $s$ -channel, and the  $u$ -channel.

The higher derivative operators in (1) determine the coefficient of  $O(s^2)$  terms in the Taylor expansion of the amplitudes around  $s = 0$ . These are extracted by a twice subtracted dispersion relation

$$i\pi\mathcal{A}''(s=0) = \oint_{|s|=\Lambda^2} \frac{ds}{s^3} \mathcal{A}(s) + \gamma \int_{m^2}^{\Lambda^2} \frac{ds}{s^3} \text{Disc}\mathcal{A}(s), \quad (50)$$

where  $\mathcal{A}$  can be  $\mathcal{A}_{h\rightarrow h}$ ,  $\mathcal{A}_{h\rightarrow AA} + \mathcal{A}_{AA\rightarrow h}$ , or  $\mathcal{A}_{AA\rightarrow AA}|_{t=0}$ .  $\gamma = 1$  except for the photon 2-to-2 amplitude where one can use crossing relation  $s \leftrightarrow u$  to conclude  $\gamma = 2$  (see figure 6).

The choices for the graviton polarization tensor are the same as in the previous section, spatial symmetric tensors decomposed into one  $l = 0$ , and five polarizations of  $l = 2$ . The photon polarizations are chosen to be real, space-like and transverse. There are two independent configurations: when  $e_1$  and  $e_2$  are parallel and when they are perpendicular. Working in the basis (9), which is related to (1) via

$$c_7 = c_{F^4} - \frac{1}{2}\tilde{c}_{F^4}, \quad c_8 = \tilde{c}_{F^4}, \quad (51)$$

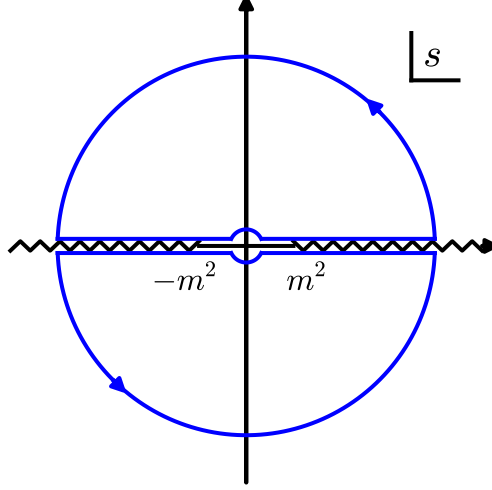


Figure 6: The complex  $s$  plane contour for  $\mathcal{A}_{AA \rightarrow AA}$  dispersion relation. Since the loops of photons are neglected at low energies the amplitude is analytic for  $|s| < m^2$ . The  $s$ -channel discontinuity along positive  $s$  and the  $u$ -channel discontinuity along negative  $s$  axis are related by crossing.

each of the two configurations in  $\mathcal{A}_{AA \rightarrow AA}$  singles out one of the two terms:

$$\mathcal{A}''_{AA \rightarrow AA}|_{s=0} = 32c_{F^4}(e_1 \cdot e_2)^2 + 8\tilde{c}_{F^4}(e_1^2 e_2^2 - (e_1 \cdot e_2)^2), \quad (52)$$

where prime denotes derivative with respect to  $s$  at constant  $t = 0$ . Finally the second derivative of the decay amplitude is given in terms of the basis

$$E_1 = (e_1 \cdot e_2)\text{Tr}[\varepsilon_{(\lambda)}], \quad E_2 = (\hat{k}_1 \cdot \varepsilon_{(\lambda)} \cdot \hat{k}_2)e_1 \cdot e_2, \quad E_3 = e_1 \cdot \varepsilon_{(\lambda)} \cdot e_2, \quad (53)$$

by

$$\mathcal{A}''_{h \rightarrow AA}|_{s=0} = \left(-4c_4 - \frac{1}{2}c_5\right) E_1 + \frac{1}{2}c_5 E_2 + (-c_5 - 2c_6) E_3, \quad (54)$$

with an identical expression for  $\mathcal{A}''_{AA \rightarrow h}|_{s=0}$ .

### 3.4 Optical Theorem and the Sum-Rules

The discontinuities of the amplitudes along the real  $s$  axis are obtained from the identity (25). They are given in terms of the Whiteman functions with a flow of time-like 4-momentum

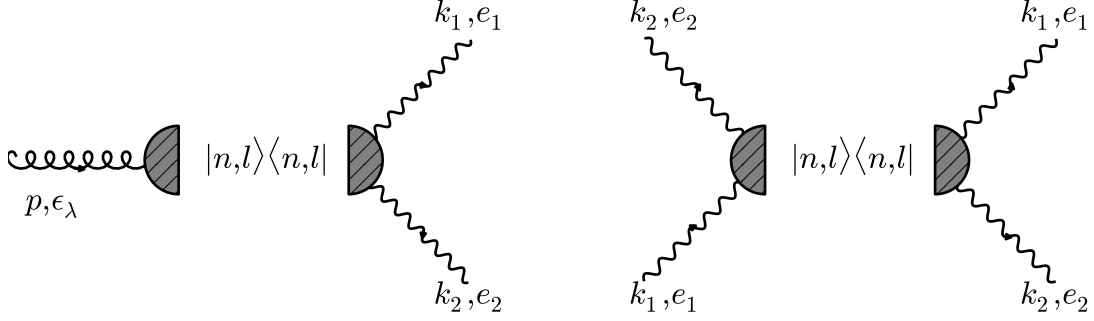


Figure 7: The discontinuities of the decay amplitude and the forward amplitude can be related to the matrix elements of the QFT states with the stress tensor and the two on-shell photons. While only  $l = 0, 2$  states contribute to the cut of the decay amplitude, the discontinuity of the forward 2-to-2 amplitude gets contribution from intermediate states with all angular momenta.

with positive energy from right to left:

$$\text{Disc}_{s>m^2}(\mathcal{A}_{h\rightarrow AA} + \mathcal{A}_{AA\rightarrow h}) = \frac{1}{2} \langle J_{e_1}(k_1) J_{e_2}(k_2) | T_\lambda(p) \rangle - c.c. \quad (55)$$

$$\text{Disc}_{s>m^2} \mathcal{A}_{AA\rightarrow AA} = i \langle J_{e_1}(k_1) J_{e_2}(k_2) | J_{e_1}(k_1) J_{e_2}(k_2) \rangle ,$$

where the “off-shell graviton”  $|T_\lambda(p)\rangle$  was defined in (28), and the 2-photon state is

$$|J_{e_1}(k_1) J_{e_2}(k_2)\rangle \equiv e_1^\mu e_2^\nu \int d^4x_1 d^4x_2 e^{ik_1 \cdot x_1 + ik_2 \cdot x_2} \text{T}[J_\mu(x_1) J_\nu(x_2)] |0\rangle. \quad (56)$$

We insert a complete set of QFT states, as in (27), in the above expressions for the discontinuities, see figure 7. In contrast to the single graviton, the 2-photon state is allowed by angular momentum conservation to have a nonzero matrix element with intermediate states of any angular momentum  $l$ . It will soon become clear that the contribution of all  $l \neq 0, 2$  states to  $c_{\text{ext}}$  will be trivially non-negative. As for the  $l = 0$  and  $l = 2$  QFT states we parametrize their matrix elements as

$$\langle J_{e_1}(k_1) J_{e_2}(k_2) | n, 0, \vec{p}_n \rangle = \frac{1}{\sqrt{3}} \beta_{n,0} (e_1 \cdot e_2) (2\pi)^4 \delta^4(p_n^\mu - p^\mu) \quad (57)$$

and

$$\langle J_{e_1}(k_1) J_{e_2}(k_2) | n, 2, m, \vec{p}_n \rangle = [\beta_{n,2} (e_1 \cdot \varepsilon_{(2,m)} \cdot e_2) + \beta'_{n,2} (\hat{k}_1 \cdot \varepsilon_{(2,m)} \cdot \hat{k}_2) (e_1 \cdot e_2)] (2\pi)^4 \delta^4(p_n^\mu - p^\mu). \quad (58)$$

Substituting (29),(30),(57) and (58) in the dispersion relations (50), summing over polarizations, and using their completeness relation (31), and comparing with the explicit IR behavior of the amplitudes (52) and (54), we obtain

$$\begin{aligned}
2ic_4 &= - \sum_{n,l=0} \frac{\alpha_{n,0}^* \beta_{n,0}}{12M_n^6} + \sum_{n,l=2} \frac{\alpha_{n,2}^* (\beta_{n,2} - 4\beta'_{n,2})}{12M_n^6} - c.c., \\
2ic_5 &= \sum_{n,l=2} \frac{2\alpha_{n,2}^* \beta'_{n,2}}{M_n^6} - c.c., \\
2ic_6 &= - \sum_{n,l=2} \frac{\alpha_{n,2}^* (2\beta'_{n,2} + \beta_{n,2})}{2M_n^6} - c.c.
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
c_{F^4} &= \sum_{n,l=0} \frac{|\beta_{n,0}|^2}{24M_n^6} + \sum_{n,l=2} \frac{2|\beta'_{n,2}|^2 + \beta_{n,2}\beta_{n,2}^* + \beta_{n,2}^*\beta'_{n,2} + 2|\beta_{n,2}|^2}{24M_n^6} + \dots, \\
\tilde{c}_{F^4} &= \sum_{n,l=2} \frac{|\beta_{n,2}|^2}{4M_n^6} + \dots
\end{aligned} \tag{60}$$

where dots include non-negative contribution from  $l \neq 0, 2$  states. Note that we have neglected the UV coefficients.

### 3.5 Truncating the Sums and Deriving the Bound

We have already seen that the sum-rule for  $c_2 + 4c_3$  is supported for the full range  $m < M_n < \Lambda$ , resulting in the logarithmically enhanced result  $2C \log(\Lambda/m)$ . On the other hand the coefficients of  $RF^2$  and  $F^4$  terms are infra-red dominated, scaling respectively as  $1/m^2$  and  $1/m^4$  as can be seen from (43). Hence we can cut off the sums at some  $\Lambda'$  with  $m \ll \Lambda' \ll \Lambda$  but  $\log(\Lambda'/m) \sim 1$ . The corrections are suppressed by powers of  $m/\Lambda' \ll 1$ .

With this truncation the particular combination  $c_5 + c_6$  that enters  $c_{\text{ext}}$  can be approximated and then bounded as

$$\begin{aligned}
|c_5 + c_6| &\simeq \left| \sum_{n,l=2}^{M_n < \Lambda'} \frac{\alpha_{n,2} (2\beta'_{n,2} - \beta_{n,2})}{4M_n^6} - c.c. \right| \\
&\leq \sqrt{\sum_{n,l=2}^{M_n < \Lambda'} \frac{|\alpha_{n,2}|^2}{4M_n^6}} \sqrt{\sum_{n,l=2}^{M_n < \Lambda'} \frac{|2\beta'_{n,2} - \beta_{n,2}|^2}{M_n^6}},
\end{aligned} \tag{61}$$

where we used the Cauchy-Schwartz inequality. The argument of the first square-root is the same as the sum (32) for the coefficient  $c_{W^2}$ , but cut off at  $\Lambda'$ . Hence the sum inside the square root is  $\simeq C \log(\Lambda'/m) \ll C \log(\Lambda/m)$ . The second square-root can be compared to the  $F^4$  contribution to  $c_{\text{ext}}$ , which is  $4c_7 + 2c_8 = 4c_{F^4}$ . The contribution of a single  $l = 2$  state to  $c_{F^4}$  sum-rule (60) can be written as

$$\frac{3|2\beta'_{n,2} - \beta_{n,2}|^2 + |4\beta'_{n,2} + 5\beta_{n,2}|^2}{336M_n^6}. \quad (62)$$

Since all other contributions are non-negative, we can bound the second square-root in (61) by  $\sqrt{112c_{F^4}}$  (recall that  $c_{F^4} > 0$ ). Therefore we obtain the bound

$$c_{\text{ext}} \geq 2C \log \frac{\Lambda}{m} - \sqrt{112c_{F^4}C \log(\Lambda'/m)} + 4c_{F^4} > 0 \quad (63)$$

as long as  $\log(\Lambda/m) \gg \log(\Lambda'/m) = O(1)$ . This completes the proof.

## 4 Conclusions

It is natural to assume that there is a QFT description with a large number of degrees of freedom and over an exponentially large range of scales below the quantum gravity scale  $\Lambda$ , and to assume that the QFT has a non-trivial  $U(1)_{EM}$  at low energies. We showed that in such a theory the leading high-derivative corrections to the Einstein-Maxwell theory are such that the charge-to-mass ratio of maximally charged black holes approaches 1 from above. Hence small maximally charged black holes are super-extremal with respect to the bigger black holes and provide them with a decay channel.

This decay channel, even though fulfilling the original motivation of the WGC, is perhaps the least interesting version of the conjecture to prove. WGC is usually envisaged as a constraint from quantum gravity on the effective field theory. Instead here it has turned out to be a consequence of the axioms of relativistic QFT. An optimistic point of view would perhaps be to interpret the result as a hint that super-extremal states might continue to exist below the Planck scale. For instance this would be a realization of the sublattice version of WGC [17] at the macroscopic level.

However there is a reason to believe that even this weakest version of the WGC goes beyond QFT unitarity and causality. Curiously, our derivation has no clear generalization to higher dimensions even though WGC itself and the question about the effect of the higher derivative operators on  $Q_{\text{ext}}/M$  both can be formulated in any number of spacetime dimensions [32]. In  $d > 4$  the coefficients of the curvature-squared operators become UV

sensitive quantities, about which one can hardly make a field theoretic statement. It is worthwhile looking more closely for field theoretic arguments in other dimensions, to better learn what there is to learn from gravity.

## Acknowledgements

I thank Giovanni Villadoro for decisive contributions and Matt Reece, Grant Remmen and Shu-Heng Shao for useful discussions.

## Appendix. The Regge intercept

The required UV behavior of the forward amplitude, namely that it grows slower than  $s^2$  at large  $s$ , would be a consequence of the Froissart bound [44] if the QFT was fully gapped. The bound follows from the exponential fall-off, as a function of the impact parameter, of interactions mediated by massive particles. This leads to an exponential suppression of partial waves with  $l \geq l_{\max} \gg \sqrt{s}/m$ , and implies that the forward amplitude grows at most as  $l_{\max}^2 \propto s$  up to logarithms.

For exchange of massless photons in the  $t$ -channel there is no exponential suppression (recall that gravitons are only external in our argument). Formally, they lead to a  $t$ -channel cut that runs all the way to  $t = 0$ . On the other hand this lower end of the cut is the relevant contribution to the scattering at large impact parameters. To illustrate the idea let us neglect photon polarizations and consider the amplitude in the impact parameter space for massless scalars. (Including external polarizations leads to a finite number of derivatives with respect to the impact parameter but doesn't change the  $s$  dependence.) This is defined as a Fourier transform of the momentum space amplitude:

$$\mathcal{A}(s, \vec{b}) = \int d^{d-2} \vec{q} e^{i\vec{q} \cdot \vec{b}} \mathcal{A}(s, t) \quad (64)$$

where  $\vec{b}$  is the impact parameter and  $\vec{q}$  the momentum transfer,  $t = -|\vec{q}|^2$ . By a contour deformation this amplitude can be related to a sum over  $t$ -channel discontinuities [45]. These discontinuities factorize into the product of the on-shell amplitudes  $\mathcal{A}_{13,t}$  and  $\mathcal{A}_{24,t}$ , which were encountered above in section 3.2. The contribution of discontinuities at  $t > m^2$  is exponentially small for  $b \gg 1/m$ . This explains why the large- $b$  amplitude effectively captures the lower end of the  $t$ -channel cut.

Furthermore, causality constrains the amplitude at fixed  $b$  to grow slower than  $s^2$  [45]. The bound is saturated in Einstein gravity where the amplitude grows as  $s^2$ . In our QFT

(when gravity is decoupled) we expect

$$\lim_{s \rightarrow \infty} \frac{1}{s^{2+\epsilon}} \int_{b=1/m}^{\infty} d^{d-2} \vec{b} \mathcal{A}(s, \vec{b}) = 0, \quad \epsilon > 0 \quad (65)$$

since the integral is convergent in the upper limit  $b \rightarrow \infty$  due to the good IR behavior of the effective theory of photons. Namely the fact, encountered in section 3.2, that  $\mathcal{A}_{13,t}$  and  $\mathcal{A}_{24,t}$  for any number of  $t$ -channel exchange photons vanish in the  $t \rightarrow 0$  limit. Assuming that the QFT amplitude grows strictly slower than  $s^2$ , i.e.  $\epsilon$  can be set to 0 in the above formula, we can give a Froissart-type argument to show the desired UV behavior. Projecting onto partial waves

$$a_l(s) = \frac{1}{\pi} \int_{-1}^1 dx P_l(x) \int_0^{\infty} d^{d-2} \vec{b} e^{-i\vec{q}\vec{b}} \mathcal{A}(s, \vec{b}), \quad t = \frac{s}{2}(x-1), \quad (66)$$

we see that the large  $l$  partial waves are suppressed as  $(\sqrt{s}b)^l$  in the lower end of the  $b$  integral. Hence assuming power-law behavior of the amplitude at large  $s$  there is some  $L \gg \frac{\sqrt{s}}{m} \log s$  such that for  $l \geq L$  we can approximate the  $b$  integral by putting a lower cutoff  $b_{\min} = 1/m$ .

As in the derivation of the Froissart bound, we can bound the forward amplitude by assuming that for all  $l < L$  the partial waves obtain their maximal value allowed by unitarity:

$$\begin{aligned} |\mathcal{A}(s, 0)| &< \frac{\pi}{2} \sum_{l=0}^L (2l+1) |P_l(1)| + \frac{\pi}{2} \left| \sum_{l=L+1}^{\infty} a_l(s) P_l(1) \right| \\ &\leq \frac{\pi}{2} L^2 + \left| \int_{1/m}^{\infty} d^{d-2} \vec{b} \mathcal{A}(s, \vec{b}) \right|. \end{aligned} \quad (67)$$

The first term grows with  $s$  as  $L^2 \propto s \log^2 s$ , and the second slower than  $s^2$ .

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